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Construction of a Time Consistent Core in Multichoice Multistage Games

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Abstract. We introduce multichoice multistage game (MMG) with perfect information and finite length in the paper. A method of construction of MMG characteristic function is proposed. The problem of time consistency and strongly time consistency (STC) of optimality principles in MMG is investigated. A necessary and sufficient condition of STC of the core in MMG with the terminal payoff function is stated. A regularization procedure leading to STC core is considered for MMG with the integral payoff function.

Key words: multichoice multistage game, time consistency, strong time consistency, core.

1 Introduction.

It is known that in the classical cooperative game every player has only next choices: to join or not into this or that coalition and solution of cooperative games is concluded with finding of grand coalition payoff sharing admissibled by every player. Chih-Ru Hsiao and Raghavan TES [1] considered multichoice games in which every player had got more than two levels for activity and proved axiomatical approach for the Shaply value in the introduced game class. Further A. van den Nouweland, S. Tijs, J. Potters, J. Zarzuelo [2] continued researching of transferring possibility of solution conceptions for cooperative games to multichoice one. In particular the core was investigated. Nevertheless the mentioned papers related to the static games. This paper we try to determine multichoice multistage game (MMG) and concern the problem of core time inconsistency.

In the contemporary life we can find many examples of a coalition creation for a finite time, when players enter into the coalition unsimultaneously, and jointed themselves they stays in the coalition up to the end of the setted term. For instance, the new nuclear weapons testing prohibition or the mines production prohibition are actual. In the paper we try to model similar processes by means of multi-level coalition.

2 Notations.

Let $N = \{1, \dots, n\}$ be the set of players, $K(x_0)$ – a tree-like graph with the initial node x_0. Determine MMG $\Gamma(x_0, T)$ with perfect information on $K(x_0)$, and with the length $T+1$ stages. The stage $T+1$ is the terminal one. Let players move with an order and only once within a stage. The ordering of players is not changed for the game. So any path $\bar{x} = (x_0, \dots, x)$, $x \in F(x_0)$, where $F(x_0)$ is the set of the terminal nodes of the tree $K(x_0)$, has the same length. Let $H: F(x_0) \to R^n_+$ be a payoff function, where $H(x) = (H_1(x), \dots, H_n(x))$ and $H_i(x)$ is the payoff of a player $i \in N$.

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In [1] the authors proposed to represent coalition as a non-negative integer vector $s = (s_1, \dots, s_n)$ in Z^n_+. The components show cooperation levels and belong to a finite set $M_0 = \{0, 1, \dots, m\}$, $m \geq 2$ of cooperation levels. Zero means that the corresponding player does not participate in coalition. Thus for the static cooperative game (v, N), where $v: 2^N \to R^1$, we would have $m = 1$.

Let players have got one and the same set of cooperation levels in $\Gamma(x_0, T)$. In this case the set $\overline{M}_0 = \underbrace{M_0 \times \dots \times M_0}_{n}$ is the set of all coalitions.

3 Path construction.

Suppose that for the game $\Gamma(x_0, T)$ a set of players, called a *carrier*, $R \subset N$ is generated. Let players can enter into R on any stage of the game, but once entering are not able to leave it. We propose that the cooperation level of a player is time while he is in R. Hence, a coalition s shows how long every player stays in R. To describe the current membership of R, denote $R(s,t)$ as a carrier R with a coalition s on the beginning of a stage t. Thus, $\cup_{\tau=0}^{t} R(s,\tau) \subset R(s,t)$ for every stage t. $R(s,t)$ may be interpreted as a group of players which pledged themselves to carry out some agreement for $T - t + 1$ stages. The player's loyality to the agreement is showed by the cooperation level.

We introduce one more restriction: the players in $N \setminus R(s,t)$ can not make coalitions each other. Thus for each stage t there exist $|N \setminus R(s,t)| + 1$ coalitions in the game $\Gamma(x_0, T)$. Now we define the consept of carrier formally.

Definition. *Given a coalition $s = (s_1, \dots, s_n) \in \overline{M}_0$, we shall say that a player $i \in N$ belongs to carrier R in the stage t, if $s_i \geq T - t + 1$:*

$$R(s,t) = \{i \mid i \in N, s_i \geq T - t + 1\}, \quad 0 \leq t \leq T,\ s \in \overline{M}_0.$$

Take a non-empty coalition $s \geq 0$ and consider construction of the carrier R. First we should discriminate players with the same cooperation levels. Create sets $Q(s_i)$, $i \in N$:

$$Q(s_i) = \{j \mid j \in N, s_j = s_i, s_j \neq s_k, k > i\}.$$

In general, $Q(s_i)$ may be empty. If $Q(s_i)$ is non-empty, then it includes all players having the same cooperation level and i – the largest index of such players.

By excluding empty set $Q(s_i)$ for $i \in N$, we arrange and relavel the rest sets in order of increasing of s_i. We get a sequence $Q(s_{i_1}), Q(s_{i_2}), \dots, Q(s_{i_{k^*}})$, where $s_{i_1} < s_{i_2} < \dots < s_{i_{k^*}}$ for $k^* = 1, \dots, n$, and k^* is the number of non-empty sets $Q(\cdot)$. Hence, we have got a partition of N

$$1)\ \bigcup_{j=1}^{k^*} Q(s_{i_j}) = N$$
$$2)\ \forall j', j'' = 1, \dots, k^* \quad Q(s_{i_{j'}}) \cap Q(s_{i_{j''}}) = \emptyset.$$

If $k^* = n$, then each $Q(\cdot)$ is a singlton set which consists of one player, i.e., $Q(s_{i_j}) = \{i_j\}$ for $j = 1, \dots, n$.

We distinguish two types of coalitions.

Case 1. Suppose there is not a player who chose the maximal cooperation level in the coalition $s \geq 0$. Then $s_{i_{k^*}} < T + 1$ and the following construction process of carrier R occurs.

When $0 \leq t \leq T - s_{i_{k^*}}$, $R(s,t) = \emptyset$. As players in $Q(s_{i_{k^*}})$ have the maximal time period of staying in R, they are to be the first who join to R:

$$R(s, T - s_{i_{k^*}} + 1) = Q(s_{i_{k^*}}).$$

In what follows, the carrier doesn't change up to the stage $T - s_{i_{k^*-1}} + 1$, i.e.,

$$R(s,t) = R(s, T - s_{i_{k^*}} + 1), \quad T - s_{i_{k^*}} + 1 \leq t \leq T - s_{i_{k^*-1}}.$$

Players with the cooperation level $s_{i_{k^*-1}}$ enter into R on the stage $T - s_{i_{k^*-1}} + 1$:

$$R(s, T - s_{i_{k^*-1}} + 1) = Q(s_{i_{k^*}}) \cup Q(s_{i_{k^*-1}}).$$

For each stage t with $T - s_{i_{k^*-1}} + 1 \leq t \leq T - s_{i_{k^*-2}}$, the carrier R doesn't increase. The next changing of the carrier R will be on the stage $T - s_{i_{k^*-2}} + 1$:

$$R(s, T - s_{i_{k^*-2}} + 1) = Q(s_{i_{k^*}}) \cup Q(s_{i_{k^*-1}}) \cup Q(s_{i_{k^*-2}}).$$

The same argument is demonstrated in each remainder stages. Finally, we come to the last situation:

a) when $s_{i_1} > 0$, then $R = N$ on and after the stage $T - s_{i_1} + 1$;

b) when $s_{i_1} = 0$, the players in $Q(s_{i_1})$ will not enter into R up to the end of the game $\Gamma(x_0, T)$ and the constructing of the carrier R will be finished on the stage $T - s_{i_2} + 1$:

$$R(s,t) = N \setminus Q(s_{i_1}), \quad T - s_{i_2} + 1 \leq t \leq T.$$

Case 2. Let $s_{i_{k^*}} = T + 1$ now. It means that there exist players whose time of being in R is $T + 1$ stages. To put in another way, the carrier R isn't empty on the original stage of the game $\Gamma(x_0, T)$ yet:

$$R(s,t) = Q(s_{i_{k^*}}), \quad 0 \leq t \leq T - s_{i_{k^*-1}}.$$

The further development of R is similar with the case $s_{i_{k^*}} < T + 1$, i.e., the carrier R increases on the "crucial" stages $T - s_{i_j} + 1$, $j = 1, \ldots, k^*$, and doesn't change on the other one.

As R increases monotonically, the set of the players decreases if R is considered as one player. We need to introduce two additional sets:

1) $Y(t)$ – a set of nodes of the tree $K(x_0)$, such that the game $\Gamma(x_0, T)$ is able to occur in these nodes after $t - 1$ stages;

2) $N(t)$ – a set of players on the beginning of a stage t. For instance, let $N \setminus R(s,t) = \{i_1, \ldots, i_l\}$ and let $R(s,t)$ be considered as one player, then $N(t) = \{i_1, \ldots, i_l, R(s,t)\}$.

We are going to find pathes of the game development using the Nash solution. For the path to be unique for every coalition we will apply the following algorithm of a solution selection. Let A be the set of the Nash solutions in the game $\Gamma(x_0, T)$. If $|A| = 1$ then we will get an unique path, so let $|A| > 1$. In this case we take a subset $A_0 \subset A$ of the Nash solutions maximizing the common payoff. If the restriction is satisfied by only one solution, i.e., $|A_0|$=1, we will get an unique corresponding path. Otherwise we consider a subset $A_1 \subset A_0$ of the Nash solutions that maximize the common payoff and the payoff of the first player. Again if $|A_1| = 1$, we will get an unique path. Otherwise we consider a subset $A_2 \subset A$ of the Nash solutions that maximize the common payoff, the first player's payoff and the second player's payoff. If $|A_2| = 1$, we will get an unique path and so on. We can continue the procedure up to the n-th player. If $|A_n| > 1$ then the needed Nash solution is arbitrary taken from A_n.

We find the path related to the coalition s moving from the terminal nodes to the initial one and looking for temporary Nash solutions on parts of the way where the carrier is constant. For the sake of simplicity, first we investigate a case when s hasn't got players with the same cooperation levels. Of course it demands $T + 1 \geq n$.

Let $s_{i_1} > 0$, then $R = N$ for $T - s_{i_1} + 1 \leq t \leq T$ and for every node $y^{T-s_{i_1}+1} \in Y(T - s_{i_1} + 1)$ in subtrees $K(y^{T-s_{i_1}+1})$, players realize trajectories $\{y^{T-s_{i_1}+1}, \ldots, \overline{y}(y^{T-s_{i_1}+1})\}$, such that

$$\sum_{i \in N} H_i(\overline{y}(y^{T-s_{i_1}+1})) = \max_{x \in F(y^{T-s_{i_1}+1})} \sum_{i \in N} H_i(x).$$

When $T - s_{i_2} + 1 \leq t \leq T - s_{i_1}$

$$N(t) = \{i_1, R(s, T - s_{i_2} + 1)\}.$$

To determine behaviour of player i_1 on the stages $T - s_{i_2} + 1 \le t \le T - s_{i_1}$, we have to know what part of the common payoff $\sum_{i \in N} H_i(\overline{y}(y^{T-s_{i_1}+1}))$ is got by the player i_1 for every node $y^{T-s_{i_1}+1} \in Y(T - s_{i_1} + 1)$. For this reason we consider cooperative positional subgames $\widetilde{\Gamma}(y^{T-s_{i_1}+1}, s_{i_1})$ on subtrees $K(y^{T-s_{i_1}+1})$, where $y^{T-s_{i_1}+1} \in Y(T - s_{i_1} + 1)$ with the length s_{i_1} stages, the players set N and the characteristic function $w(P, y^{T-s_{i_1}+1}, s_{i_1})$, $P \subset N$. We suppose that core $C(y^{T-s_{i_1}+1}, s_{i_1})$ is a solution of the $\widetilde{\Gamma}(y^{T-s_{i_1}+1}, s_{i_1})$, $y^{T-s_{i_1}+1} \in Y(T - s_{i_1} + 1)$ (let $C(y^{T-s_{i_1}+1}, s_{i_1})$ be non-empty). For every subgame $\widetilde{\Gamma}(y^{T-s_{i_1}+1}, s_{i_1})$, an optimal imputation $\eta(y^{T-s_{i_1}+1})$ is taken and fixed. Then from $T - s_{i_2} + 1$ to $T - s_{i_1}$ stages, players i_1 and $R(s, T - s_{i_2} + 1)$ base themselves on the payoffs $\eta_{i_1}(y^{T-s_{i_1}+1})$ and $\sum_{j \in R(s, T - s_{i_2}+1)} \eta_j(y^{T-s_{i_1}+1})$, respectively, when they take decisions. If we find the Nash solution $z^{T-s_{i_1}+1}(y^{T-s_{i_2}+1}) \in Y(T - s_{i_1} + 1) \bigcap K(y^{T-s_{i_2}+1})$, for $N(y^{T-s_{i_2}+1}) = \{i_1, R(s, T - s_{i_2} + 1)\}$ and every node $y^{T-s_{i_2}+1} \in Y(T - s_{i_2} + 1)$, we will know the path of the party after $T - s_{i_2}$ stage:

$$\left\{ y^{T-s_{i_2}+1}, \ldots, \overline{y}(z^{T-s_{i_1}+1}(y^{T-s_{i_2}+1})) \right\}.$$

Hence, an imputation $\eta(z^{T-s_{i_1}+1}(y^{T-s_{i_2}+1}))$ is stated in accordance with a node $y^{T-s_{i_2}+1} \in Y(T - s_{i_2} + 1)$.

Knowing payoffs which players get if the trajectory goes through this or that node of the set $Y(T - s_{i_2} + 1)$ leads us to the Nash solution construction for $N(T - s_{i_3} + 1) = \{i_1, i_2, R(s, T - s_{i_3} + 1)\}$. Let $z^{T-s_{i_2}+1}(y^{T-s_{i_3}+1}) \in Y(T - s_{i_2} + 1) \cap K(y^{T-s_{i_3}+1})$ be the Nash solution for $N(T - s_{i_3} + 1)$, then after stage $T - s_{i_3}$ the path corresponding to the coalition s is

$$\left\{ y^{T-s_{i_3}+1}, \ldots, \overline{y}(z^{T-s_{i_1}+1}(z^{T-s_{i_2}+1}(y^{T-s_{i_3}+1}))) \right\}, \quad y^{T-s_{i_3}+1} \in Y(T - s_{i_3} + 1),$$

and payoffs based on players from $N(T - s_{i_4} + 1)$ are determined by imputations

$$\eta(z^{T-s_{i_1}+1}(z^{T-s_{i_2}+1}(y^{T-s_{i_3}+1}))).$$

Continuing in the same way, we shall reach the initial node x_0 ultimately and get the path x_s of the coalition s:

$$x_s = \left\{ x_0, \ldots, \overline{y}(z^{T-s_{i_1}+1}(\ldots(z^{T-s_{i_{k^*}}+1}(x_0))\ldots)) \right\}.$$

If $s_{i_1} = 0$, then the payoff of the player i_1 is known yet – it is the terminal payoff $H_{i_1}(\cdot)$. Therefore we don't have any obstacles to construct the Nash solution $z^T(y^{T-s_{i_2}+1}) \in Y(T) \bigcap K(y^{T-s_{i_2}+1})$ for players in $N(y^{T-s_{i_2}+1}) = \{i_1, R(s, T - s_{i_2} + 1)\}$. However, in order to find the Nash solution for players in $N(y^{T-s_{i_3}+1}) = \{i_1, i_2, R(s, T - s_{i_3} + 1)\}$ on stages $T - s_{i_3} + 1 \le t \le T - s_{i_2}$, we need to know the payoff of the player i_2 when he participates in $R(s, t)$, $T - s_{i_2} + 1 \le t \le T$. For the sake of the reason we deal with cooperative positional subgames $\widetilde{\widetilde{\Gamma}}(y^{T-s_{i_2}+1}, s_{i_2})$ on subtrees $K(y^{T-s_{i_2}+1})$, $y^{T-s_{i_2}+1} \in Y(T - s_{i_2} + 1)$ with the players set $N \setminus \{i_1\}$, and the length s_{i_2} stages. The characteristic function $\widetilde{\widetilde{w}}(P, y^{T-s_{i_2}+1}, s_{i_2})$ of the subgame $\widetilde{\widetilde{\Gamma}}(y^{T-s_{i_2}+1}, s_{i_2})$ is constructed by means of the characteristic function $\widetilde{w}(P, y^{T-s_{i_2}+1}, s_{i_2})$ of the subgame $\widetilde{\Gamma}(y^{T-s_{i_2}+1}, s_{i_2})$ (the case $s_{i_1} > 0$):

$$\widetilde{\widetilde{w}}(P, y^{T-s_{i_2}+1}, s_{i_2}) = \begin{cases} \widetilde{w}(P, y^{T-s_{i_2}+1}, s_{i_2}) \dfrac{\sum_{j \in N \setminus \{i_1\}} H_j(z^T(y^{T-s_{i_2}+1}))}{\max_{x \in F(y^{T-s_{i_2}+1})} \sum_{j \in N} H_j(x)}, & P \neq N \setminus \{i_1\}, \\ & P \subset N \setminus \{i_1\} \\ \sum_{j \in N \setminus \{i_1\}} H_j(z^T(y^{T-s_{i_2}+1})), & P = N \setminus \{i_1\}. \end{cases}$$

Deal with the formula for $P \neq N \setminus \{i_1\}$. Here $\tilde{\tilde{w}}(P, y^{T-s_{i_2}+1}, s_{i_2})$ is proportional decreasing of $\tilde{w}(P, y^{T-s_{i_2}+1}, s_{i_2})$, and the factor of proportionality taken from relation of the payoff value of the $R(s, T-s_{i_2}+1)$ to one which he could have get if he had included all players. Let $\tilde{\tilde{C}}(y^{T-s_{i_2}+1}, s_{i_2})$ be the core of a subgame $\tilde{\tilde{\Gamma}}(y^{T-s_{i_2}+1}, s_{i_2})$. We choose one optimal imputation $\eta(y^{T-s_{i_2}+1}) \in \tilde{\tilde{C}}(y^{T-s_{i_2}+1}, s_{i_2}) \subset R_+^{n-1}$ for every node $y^{T-s_{i_2}+1} \in Y(T-s_{i_2}+1)$ and fix him. Hence n – dimensional payoff-vectors $(\eta(y^{T-s_{i_2}+1}), H_{i_1}(y^{T-s_{i_2}+1}))$ are in agreement with nodes from $Y(T-s_{i_2}+1)$. Thus we can find the Nash solution between players in $N(T-s_{i_3}+1)$ on a set $Y(T-s_{i_2}+1) \bigcap K(y^{T-s_{i_3}+1})$ of a subtree $K(y^{T-s_{i_3}+1})$ for every $y^{T-s_{i_3}+1} \in Y(T-s_{i_3}+1)$. In what follows trajectory developments are analogous to the case $s_{i_1} > 0$.

We are able to use the concerned algorithm when there are players with the same cooperation levels in $s = (s_1, \dots, s_n)$ as well. Of course we have to take account of that sets $N(T-s_{i_{j+1}}+1)$, $s_{i_{j+1}}$ with $|Q(s_{i_j})| > 1$ consist of more number of players:

$$N(T-s_{i_{j+1}}+1) = N(T-s_{i_j}+1) \bigcup Q(s_{i_j}).$$

4 Characteristic function.

Definition. *We shall say that a function* $v(s, x_0, T) = \sum_{i \in N} H_i(x_s) \frac{s_i}{T+1}$, $s \in \overline{M}_0$ *is called the characteristic function of the game* $\Gamma(x_0, T)$, *if it is superadditive, i.e.,* $\forall \overline{s}, \overline{\overline{s}} \in \overline{M}_0$ *such that for every player* i *with* $\overline{s}_i \cdot \overline{\overline{s}}_i = 0$,

$$v(\overline{s}, x_0, T) + v(\overline{\overline{s}}, x_0, T) \leq v(\overline{s} \cup \overline{\overline{s}}, x_0, T).$$

The coefficient $\frac{s_i}{T+1}$ may be explained as a tax for uncooperation. If a player i in coalition s stays on the level 0, he gives nothing for the benefit of the s.

Definition. *We shall say that the characteristic function* $v(s, x_0, t)$ *of the game* $\Gamma(x_0, T)$ *is non-decreasing if for every* $i \in N$, *and every* s *with* $s_i \geq 1$,

$$v(s, x_0, T) \geq v(s \| (s_i - 1), x_0, T),$$

where $s \| (s_i - 1) = (s_1, \dots, s_{i-1}, s_i - 1, s_{i+1}, \dots, s_n)$.

We confine ourself to dealing with multichoice multistage games with non-decreasing characteristic functions.

5 Set of imputations and core.

Let $L^0 = \{(i, s_i) \mid i \in N, s_i \in M_0\}$, and let $\Delta^0 \colon L^0 \to R_+^1$ be a mapping such that

$$\Delta^0_{is_i} \geq v((0, \dots, 0, s_i, 0, \dots, 0), x_0, T) - v((0, \dots, 0, s_i - 1, 0, \dots, 0), x_0, T), \quad s_i \in M_0.$$

Definition. *We shall say that matrix* $\xi^0 = \{\Delta^0_{ij}\}$, $i = \overline{1,n}$, $j = \overline{1,m}$ *is a division of the game* $\Gamma(x_0, T)$, *if the following restrictions are satisfied*

1) $\xi^0(0, \dots, 0, s_i, 0, \dots, 0) \geq v((0, \dots, 0, s_i, 0, \dots, 0), x_0, T), \quad \forall i \in N, \forall s_i \in M_0.$

2) $\xi^0(m, \dots, m) = v((m, \dots, m), x_0, T), \quad$ *where* $\xi^0(s) = \sum_{i \in N} \sum_{j=0}^{s_i} \Delta^0_{ij}.$

We denote the set of imputations of the game $\Gamma(x_0, T)$ by $I(x_0, T)$.

Definition. *A set* $C(x_0, T)$ *of imputations such that*

$$C(x_0, T) = \{\xi^0 \mid 1) \xi^0 \in I(x_0, T);\ 2) \forall s \in \overline{M}_0, \xi^0(s) \geq v(s, x_0, T)\}$$

is called by the core of the game $\Gamma(x_0, T)$.

6 Time consistency.

we see that players are interested in choosing of the maximal cooperation level for the trajectory $\overline{x}$ maximizing the grand coalition payoff

$$\sum_{i \in N} H_i(\overline{x}) = \max_{x \in F(x_0)} \sum_{i \in N} H_i(x)$$

to realize. Call $\overline{x}$ *an optimal path* and consider along it subgames $\Gamma(\bar{x}^t, T-t)$, $1 \leq t \leq T$. Introduce the following notations for a subgame $\Gamma(\bar{x}^t, T-t)$:

1) $M_t = \{0, 1, \ldots, m-t\}$ – the cooperation levels set;
2) $\overline{M}_t = \underbrace{M_t \times \cdots \times M_t}_{n}$ – the coalition set;
3) $v(s^t, \bar{x}^t, T-t)$ – the characteristic function, $s^t \in \overline{M}_t$;
4) $I(\bar{x}^t, T-t)$ – the set of imputations;
5) $C(\bar{x}^t, T-t)$ – the core.
6) let $L^t = \{(i, s_i^t) \mid i \in N, S_i^t \in M_t\}$.

Definition. *A division $\xi^0 \in C(x_0, T)$ of the game $\Gamma(x_0, T)$ with the terminal payoff function is called time consistent, if for every stage t the matrix $\overline{\xi} = \{\Delta_{ij}^0\}$, $(i,j) \in L^t$ is an optimal imputation of the subgame $\Gamma(\bar{x}^t, T-t)$.*

Definition. *We shall say that $C(x_0, T)$ is time consistent optimality principle (TCOP), if every imputation $\xi^0 \in C(x_0, T)$ is time consistent.*

In [3] a condition of time consistency for cooperative multistage games was stated. We prove an analogous one for multichoice multistage games.

Proposition. *The core $C(x_0, T)$ of the game $\Gamma(x_0, T)$ with the terminal payoff function is a TCOP if and only if for every stage t the core $C(\bar{x}^t, T-t)$ of the subgame $\Gamma(\bar{x}^t, T-t)$ consists of an unique imputation ξ^t, where*

$$\xi^t = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ H_1(\overline{x}) & \cdots & H_n(\overline{x}) \end{pmatrix}$$

Proof. The core $C(\overline{x}^T, 0)$ of the subgame $\Gamma(\overline{x}^T, 0)$ consists of an unique imputation

$$\xi^T = \begin{pmatrix} 0 & \cdots & 0 \\ H_1(\overline{x}) & \cdots & H_n(\overline{x}) \end{pmatrix}.$$

Therefore $\Delta_{i0}^T = H_i(\overline{x}^T)$, $i \in N$. At the same time for every stage t

$$v(m, x_0, T) = v((m-t)^t, \bar{x}^t, T-t) = \sum_{i \in N} H_i(\overline{x}^T),$$

where $(m-t)^t = (m-t, \ldots, m-t) \in R^n$. It means that these equalities are able to be satisfied only by an imputation $\xi^0 \in C(x_0, T)$: $\Delta_{i0}^0 = H_i(\overline{x}^T)$, and $\forall j \in M_0 \setminus \{0\}$, $\Delta_{ij}^0 = 0$. If such imputation ξ^0 doesn't belong to $C(x_0, T)$, then the imputation $\overline{\xi} = \{\Delta_{ij}^0\}_{(i,j) \in L^T} \notin C(\overline{x}^T, 0)$ and $C(x_0, T)$ is time inconsistent. Being obviously to backwards, the proof of the proposition is complete.

It is known that regularization of time inconsistent optimality principles is impossible if the payoff function is terminal. Let us consider MMG with integral payoff functions. Let

$$H_i(x) = \sum_{t=0}^{T} h_i(x), \quad h_i(x) \geq 0, i \in N.$$

and Δ^0 be distributed in time as well:

$$\Delta^0_{ij} = \sum_{t=0}^{T} \Delta^0_{ij}(t), \quad \Delta^0_{ij}(t) \geq 0.$$

Definition. *An imputation* $\xi^0 = \{\Delta^0_{ij}\}$ *of the* $\Gamma(x_0, T)$ *with the integral payoff function is called time consistent if for every stage* θ

$$\overline{\xi}^\theta = \{\overline{\Delta}^\theta_{ij}\} \in C(\overline{x}^\theta, T-\theta), \quad where\ \overline{\Delta}^\theta_{ij} = \sum_{t=\theta}^{T} \Delta^0_{ij}(t),\ (i,j) \in L^\theta.$$

Definition. *The core* $C(x_0, T)$ *of the game* $\Gamma(x_0, T)$ *with an integral payoff function is called TCOP if every imputation* $\xi^0 \in C(x_0, T)$ *is time consistent.*

Definition. *An imputation* $\xi^0 = \{\Delta^0_{ij}\} \in C(x_0, T)$ *is called strongly time consistent, if for every stage* θ *and every imputation* $\xi^{\theta+1} = \{\Delta^{\theta+1}_{ij}\} \in C(\overline{x}^{\theta+1}, T-\theta-1)$ *the matrix* $\overline{\xi} = \{\overline{\Delta}_{ij}\}$*, where*

$$\overline{\Delta}_{ij} = \begin{cases} \sum\limits_{t=0}^{\theta} \Delta^0_{ij}(t) + \sum\limits_{t=\theta+1}^{T} \Delta^{\theta+1}_{ij}(t), & (i,j) \in L^{\theta+1} \\ \sum\limits_{t=0}^{\theta} \Delta^0_{ij}(t), & (i,j) \in L^0 \setminus L^{\theta+1} \end{cases}$$

belongs to $C(x_0, T)$

Definition. $C(x_0, T)$ *is called strongly time consistent optimality principle (STCOP) if every imputation* $\xi^0 \in C(x_0, T)$ *is strongly time consistent.*

It is clear that, as the cooperative multistage game being a special case of the multichoice multistage game, the problem on time inconsistency of the classical optimality principles is actual in multichoice multistage games as well. We propose a procedure of regularization of a time inconsistent core.

7 Regularization.

For the game $\Gamma(x_0, T)$, we construct a new (regularized) game $\overline{\Gamma}(x_0, T)$ with the strongly time consistent core $\overline{C}(x_0, T)$. Let the payoff function $H(\bar{x})$ is additively separable over the stages:

$$H_i(\bar{x}) = \sum_{t=0}^{T} h_i(\bar{x}^t), \quad h_i(\cdot) \geq 0.$$

For the simplicity of the notification, let $\min\{s^t, m^\theta\}$, $\theta \leq t$, denote a coalition

$$(\min\{s^t_1, m^\theta\}, \min\{s^t_2, m^\theta\}, \ldots, \min\{s^t_n, m^\theta\}).$$

Introduce functions $w(s^t, t)$, $0 \leq t \leq T$ as follows:

$$w(s^t, t) = \frac{1}{T-t+1}\Bigg((T-t+1)\sum_{i\in N} h_i(\bar{x}^t) -$$

$$\sum_{\theta=t}^{T} \frac{(v(m^\theta, \bar{x}^\theta, T-\theta) - v(\min\{s^t, m^\theta\}, \bar{x}^\theta, T-\theta)) \sum_{i\in N} h_i(\bar{x}^\theta)}{v(m^\theta, \bar{x}^\theta, T-\theta)}\Bigg),$$

or

$$w(s^t,t) = \frac{\sum_{i\in N} h_i(\bar{x}^t)}{T-t+1} \sum_{\theta=t}^{T} \frac{v(\min\{s^t,m^\theta\},\bar{x}^\theta,T-\theta)}{v(m^\theta,\bar{x}^\theta,T-\theta)}.$$

We can see every function $w(s^t,t)$, $0 \le t \le T$ is superadditive over coalition:

$$w(m^t,t) = \sum_{i\in N} h_i(\bar{x}^t),$$

and

$$w(0,t) = 0.$$

Hence the functions $w(s^t,t)$ may serve by distribution of a characteristic function of a game $\overline{\Gamma}(x_0,T)$

$$\bar{v}(s^0,x_0,T) = \sum_{\tau=0}^{T} w(\min\{s^0,m^\tau\},\tau).$$

Note that for every t and s^t the function $w(s^t,t)$ is non-negative. That is why the characteristic functions of subgames $\overline{\Gamma}(\bar{x}^t,T-t)$:

$$\bar{v}(s^t,\bar{x}^t,T-t) = \sum_{\tau=t}^{T} w(\min\{s^t,m^\tau\},\tau)$$

don't increase by stages. Let $\xi^t = \{\Delta^t_{ik}\}$, $0 \le t \le T$ be arbitrary taken optimal imputations in the subgames $\Gamma(\bar{x}^t,T-t)$. For every stage t substitute these imputations into the functions $w(s^t,t)$ in place of $v(\min\{s^t,m^\theta\},\bar{x}^t,T-t)$. Taking into account the determination of $\bar{v}(s^0,x_0,T)$, we have got an optimal imputation $\overline{\xi}^0 = \{\overline{\Delta}^0_{ij}\}$ of the game $\overline{\Gamma}(x_0,T)$:

$$\sum_{i\in N}\sum_{j=0}^{s^0_i} \overline{\Delta}^t_{ij} = \frac{\sum_{i\in N} h_i(\bar{x}^t)}{T+1} \sum_{\theta=0}^{T} \frac{\sum_{i\in N} \sum_{k=0}^{(\min\{s^0,m^\theta\})_i} \Delta^\theta_{ik}}{v(m^\theta,\bar{x}^\theta,T-\theta)}.$$

Consideration of the determination of the functions $\overline{v}(s^t,x^t,T-t)$ leads us to conclusion that imputations $\overline{\xi}^t = \{\overline{\Delta}^0_{ij}\}_{(i,j)\in L^t}$ are optimal in the subgames $\overline{\Gamma}(\bar{x}^t,T-t)$. Hence the imputation $\overline{\xi}^0$ is strongly time consistent. By choosing all possible $\xi^t \in C(\bar{x},T-t)$, $0 \le t \le T$, we construct the rest imputations of the strongly time consistent core $\overline{C}(x_0,T)$.

8 Conclusion.

We considered multichoice multistage game satisfying that players are able to enter into carrier R on any stage but can not leave it. As a matter of fact, the carrier R is an usual coalition increasing in time. The vector coalition $s \in \overline{M}_0$ shows relations of players to the usual coaltion. We have prohibited players' leaving right in this paper. It seems to be interesting if the opposite prohibition is used, i.e., suppose that at the beginning of the game all players in the carrier and they can go out from it on any stage but once leaving can not enter again into carrier.

The following remark deals with characteristic function. We proposed to use a factor of proportionality which puts zero for the zero coalition. If we assume that the zero coalition is able to get non-zero value, then we can determine the characteristic function as follows.

Definition. *We shall say that a function* $v(s,x_0,T) = \sum_{i:s_i\neq 0} H_i(x_s)$, $s \in \overline{M}_0$ *is called the characteristic function of the game* $\Gamma(x_0,T)$*, if it is superadditive.*

Finally we should notice that the suggested regularization procedure creates characteristic function which redistributes payoff along the optimal trajectory and is not needed in any outside payments. It occurs as on every stage t we use payoffs that players have got yet to this stage.

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